ISRAEL JOURNAL OF MATHEMATICS **146** (2005), 93–110

EIGENVALUES AND POWER GROWTH

BY

THOMAS RANSFORD*

Département de mathématiques et de statistique
Université Laval, Québec (QC), Canada G1K 7P4
e-mail: ransford@mat.ulaval.ca

ABSTRACT

Let X be a complex Banach space and let T be a bounded linear operator on X. Denote by $\sigma_p(T)$ the point spectrum of T and by $\mathbb{T}$ the unit circle. We investigate how the growth of the sequence $\|T^n\|$ is influenced by the size of the set $\sigma_p(T) \cap \mathbb{T}$ and by the geometry of the space X. We also prove analogous results for C_0-semigroups $(T_t)_{t\geq 0}$.

1. Introduction and statement of results

Let T be a bounded linear operator on a complex Banach space X. We write $\sigma_p(T)$ for the point spectrum of T (i.e., the set of eigenvalues of T), and $\mathbb{T}$ for the unit circle. Evidently,

$$\sigma_p(T) \cap \mathbb{T} \neq \emptyset \quad \Rightarrow \quad \|T^n\| \geq 1 \quad (n \geq 1).$$

It turns out that, for certain spaces X,

$$\sigma_p(T) \cap \mathbb{T} \text{ 'large'} \quad \Rightarrow \quad \|T^n\| \text{ 'grows'}.$$

The purpose of this article is to explore this phenomenon.

Our starting point is a theorem of Jamison [7] to the effect that, if X is separable and T is power-bounded, then $\sigma_p(T) \cap \mathbb{T}$ is countable. This result is optimal in at least two senses: $\sigma_p(T) \cap \mathbb{T}$ may be any countable set (consider

* Research partially supported by grants from NSERC, FQRNT and the Canada research chairs program.
Received May 21, 2004

a unitary diagonal operator on ℓ^2), and one cannot drop the assumption that X be separable (consider a unitary diagonal operator on $\ell^2(A)$, where A is uncountable).

However, there is also a sense in which Jamison's theorem may be improved. Turning it on its head, it says that, if X is separable and $\sigma_p(T)\cap\mathbb{T}$ is uncountable, then $\sup_{n\geq 1}\|T^n\| = \infty$. This conclusion can be strengthened.

We say that a set Z of positive integers is of **density zero** if

$$\operatorname{card}(Z\cap[1,n])/n \to 0 \quad \text{as } n\to\infty.$$

THEOREM 1.1: *Let X be a separable Banach space, and let T be an operator on X such that $\sigma_p(T)\cap\mathbb{T}$ is uncountable. Then there exists a set Z of positive integers of density zero such that*

$$\lim_{\substack{n\to\infty \\ n\notin Z}} \|T^n\| = \infty.$$

Must the whole sequence $\|T^n\|\to\infty$? We suspect not, though unfortunately we have no example (but see the remark at the end of §3). The next result sheds some further light on this question.

A finite Borel measure μ on $\mathbb{T}$ is called a **Rajchman measure** if its Fourier coefficients satisfy $\lim_{|n|\to\infty}\widehat{\mu}(n) = 0$. A subset E of $\mathbb{T}$ is said to be a **set of extended uniqueness** if $\mu(E) = 0$ for every positive Rajchman measure μ. For more on these concepts, see [12] for example.

THEOREM 1.2: *Let X be a separable Banach space, and let T be an operator on X such that $\|T^n\|\not\to\infty$ as $n\to\infty$. Then the subgroup generated by $\sigma_p(T)\cap\mathbb{T}$ is a set of extended uniqueness.*

Since Lebesgue measure is a Rajchman measure, we immediately obtain the following corollary.

COROLLARY 1.3: *Assume that X is separable and that $\sigma_p(T)\cap\mathbb{T}$ has positive Lebesgue measure. Then $\|T^n\|\to\infty$ as $n\to\infty$.* ∎

Here is a further consequence.

COROLLARY 1.4: *Assume that X is separable and that $\sigma_p(T)\cap\mathbb{T}$ is of second Baire category in $\mathbb{T}$. Then $\|T^n\|\to\infty$ as $n\to\infty$.*

Proof: Since $\sigma_p(T)\cap\mathbb{T}$ is an analytic set (see §2 below), it has the Baire property [11, Theorem 21.6]. Hence, if it is also of second Baire category in $\mathbb{T}$, then, by Pettis' lemma [11, Theorem 9.9], the subgroup it generates contains a neighbourhood of the identity and thus equals the whole of $\mathbb{T}$. The result now follows from Theorem 1.2 ∎

In view of these corollaries, one might suspect that, by assuming even more about $\sigma_p(T) \cap \mathbb{T}$, one could draw a stronger conclusion about the growth of $\|T^n\|$. In fact, this is false. Even if $\sigma_p(T) \cap \mathbb{T}$ is the whole unit circle, $\|T^n\|$ can tend to infinity arbitrarily slowly.

Example 1.5: Given a sequence $(\gamma_n)_{n\geq 1}$ such that $\gamma_n > 1$ for all n and $\lim_{n\to\infty} \gamma_n = \infty$, there exists an operator T on c_0 such that $\sigma_p(T) \cap \mathbb{T} = \mathbb{T}$ and $\|T^n\| \leq \gamma_n$ for all n.

Thus, to be able say more about the growth of $\|T^n\|$, we need to assume more about the geometry of X. To this end, we recall some terminology.

Given a Bochner-integrable function $f\colon \mathbb{T} \to X$, we define its Fourier coefficients by

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta \quad (n \in \mathbb{Z}).$$

Then $f \mapsto \widehat{f}$ is a bounded linear map from $L^1(\mathbb{T}, X)$ to $\ell^\infty(\mathbb{Z}, X)$.

Let $1 < p \leq 2$ and $1/p + 1/q = 1$. The Banach space X is said to be of **Fourier type** p if $f \mapsto \widehat{f}$ is a bounded linear map from $L^p(\mathbb{T}, X)$ into $\ell^q(\mathbb{Z}, X)$. For example, X is of Fourier type p if $X = L^p$ or L^q. Also, X is of Fourier type 2 if and only if it is isomorphic to a Hilbert space, and X is of Fourier type > 1 if and only if it is of (Rademacher) type > 1. For more information on this subject, see [5] for example.

The following result is a generalization of a theorem of Nikolski about Hilbert spaces [14, p. 239].

THEOREM 1.6: *Let $1 < p \leq 2$ and $1/p + 1/q = 1$. Let X be a separable Banach space of Fourier type p, and let T be an operator on X such that $\sigma_p(T) \cap \mathbb{T}$ is of positive Lebesgue measure. Then*

$$\sum_{n\geq 1} \|T^n\|^{-q} < \infty.$$

If $1 < p \leq 2$ and $1/p + 1/q = 1$, then ℓ^q is of Fourier type p. Thus the following example shows that the conclusion of Theorem 1.6 cannot be strengthened.

Example 1.7: Given $q \geq 1$ and a sequence $(\gamma_n)_{n\geq 1}$ such that $\gamma_n > 1$ for all n and $\sum_n \gamma_n^{-q} < \infty$, there exists an operator T on ℓ^q such that $\sigma_p(T) \cap \mathbb{T} = \mathbb{T}$ and $\|T^n\| \leq \gamma_n$ for all n.

Moreover, the hypothesis of positive Lebesgue measure cannot be weakened.

Example 1.8: Given a closed subset E of $\mathbb{T}$ of zero Lebesgue measure, there exists an operator T on ℓ^2 such that $\sigma_p(T) \cap \mathbb{T} = E$ and $\sum_{n\geq 1} \|T^n\|^{-2} = \infty$.

It is however possible to say more about the case when $\sigma_p(T) \cap \mathbb{T}$ has Lebesgue measure zero, by appealing to the theory of capacities as developed in [8, Chapter III]. We briefly recall the definitions.

A **kernel** is a non-negative, integrable, even function $\Phi: \mathbb{T} \to \mathbb{R}$ which is convex on $(0, 2\pi)$. It can be shown that, if Φ is a kernel, then its Fourier coefficients satisfy $\widehat{\Phi}(n) \geq 0$ for all $n \in \mathbb{Z}$ (see [8, p. 32, Proposition 2]). Examples of kernels include the logarithmic kernel $\Phi_0(t) = \log|1/\sin(t/2)|$ and the kernels of order α defined by $\Phi_\alpha(t) = 1/|\sin(t/2)|^\alpha$ $(0 < \alpha < 1)$.

Let Φ be a kernel. The **Φ-energy** of a Borel probability measure μ on $\mathbb{T}$ is defined by

$$I_\Phi(\mu) := \iint \Phi(t-u)d\mu(t)d\mu(u).$$

The **Φ-capacity** of a subset E of $\mathbb{T}$ is defined by

$$C_\Phi(E) := \sup 1/I_\Phi(\mu),$$

where the supremum is taken over all Borel probability measures μ carried by E. In particular, E is of positive Φ-capacity if and only if it carries a probability measure of finite Φ-energy.

We shall also need some notation from the theory of Banach spaces. Following Pisier [15, Definition 4.1], we say that a Banach space X is **θ-Hilbertian** $(0 \leq \theta \leq 1)$ if there exists an interpolation pair (Y, H) of Banach spaces such that H is a Hilbert space and $X = [Y, H]_\theta$. Here, $[\cdot,\cdot]_\theta$ denotes the standard complex interpolation method (see, e.g., [1, Chapter 4]).

For example, let $1 < p \leq 2$ and $1/p + 1/q = 1$. Then L^p and L^q are both $2/q$-Hilbertian (just take $H = L^2$ and $Y = L^1$ or L^∞ respectively). More generally, a Banach lattice is $2/q$-Hilbertian provided that it is p-convex and q-concave [15, Theorem 2.3]. Note also that, if a Banach space is $2/q$-Hilbertian, then it is necessarily of Fourier type p [5, Theorem 7.16].

Once again, the following result is a generalization (and a slight strengthening) of a theorem of Nikolski for Hilbert spaces [14, p. 239].

THEOREM 1.9: *Let $1 < p \leq 2$ and $1/p + 1/q = 1$. Let X be a separable Banach space which is $2/q$-Hilbertian. Let Φ be a kernel. Let T be operator on X such that $\sigma_p(T) \cap \mathbb{T}$ is of positive Φ-capacity. Then*

$$\sum_{n\geq 1} \widehat{\Phi}(n)^{q/2} \|T^n\|^{-q} < \infty.$$

Applying this theorem with the particular kernels Φ_0, Φ_α mentioned above, we obtain the following corollaries.

COROLLARY 1.10: *Assume that X is a separable Hilbert space, and that $\sigma_p(T) \cap \mathbb{T}$ is positive logarithmic capacity. Then*

$$\sum_{n\geq 1} n^{-1} \|T^n\|^{-2} < \infty.$$

Proof: Positive logarithmic capacity is equivalent to positive Φ_0-capacity. Also, $\widehat{\Phi}_0(n) \asymp 1/n$ (see [8, p. 40]). The result therefore follows from Theorem 1.9 with $p = q = 2$. ∎

COROLLARY 1.11: *Let $1 < p \leq 2$ and $1/p + 1/q = 1$, and let $\alpha \geq 0$. Assume that X is separable and $2/q$-Hilbertian, and that $\sigma_p(T) \cap \mathbb{T}$ is of Hausdorff dimension $> \alpha$. Then*

$$\sum_{n\geq 1} n^{(\alpha-1)q/2} \|T^n\|^{-q} < \infty.$$

Proof: By [8, p. 34, Théorème I], the fact that $\sigma_p(T) \cap \mathbb{T}$ is of Hausdorff dimension $> \alpha$ implies that it has positive Φ_α-capacity. Also, $\widehat{\Phi}_\alpha(n) \asymp n^{\alpha-1}$ (see [8, p. 40]). Thus the result again follows from Theorem 1.9. ∎

Finally, we mention an example relating to the sharpness of Corollary 1.11. It is expressed in terms of Minkowski dimension, which is in general larger than Hausdorff dimension, though for many regular sets the two dimensions coincide (for more details, see [4, Chapters 2, 3]).

Example 1.12: Given $\alpha > 0$ and a closed subset E of $\mathbb{T}$ of upper Minkowski dimension $< \alpha$, there exists an operator T on ℓ^2 such that $\sigma_p(T) \cap \mathbb{T} = E$ and $\sum_{n\geq 1} n^{\alpha-1} \|T^n\|^{-2} = \infty$.

The rest of the paper is organized as follows. In §2 we take care of certain measurability questions. The proofs of Theorems 1.1 and 1.2 are presented in §3, and those of Theorems 1.6 and 1.9 are given in §4. The various examples are constructed in §5. Finally, in §6, we state and prove some analogues of the results above in the setting of continuous semigroups of operators.

2. Measurability

Our first task is to take care of some measurability problems. We recall some basic definitions, referring the reader to [11, 16] for further details.

A subset E of $\mathbb{T}$ is **universally measurable** if it is measurable with respect to every σ-finite Borel measure on $\mathbb{T}$. The universally measurable sets form a σ-algebra, and a function defined on $\mathbb{T}$ is said to be universally measurable if it is measurable with respect to this σ-algebra.

A subset E of $\mathbb{T}$ is **analytic** (or **Souslin**) if there exists a continuous surjection from a complete separable metric space onto E. Every Borel subset of $\mathbb{T}$ is analytic, and every analytic subset of $\mathbb{T}$ is universally measurable (see, e.g., [11, Theorem 21.10]).

LEMMA 2.1: *Let X be a separable Banach space and let T be an operator on X. Then $\sigma_p(T) \cap \mathbb{T}$ is an analytic subset of $\mathbb{T}$, and there exists a universally measurable function $g\colon \mathbb{T} \to X$ such that*

$$(1) \qquad Tg(\lambda) = \lambda g(\lambda) \ (\lambda \in \mathbb{T}) \quad \textit{and} \quad \|g(\lambda)\| = \begin{cases} 1, & \lambda \in \sigma_p(T) \cap \mathbb{T}, \\ 0, & \lambda \notin \sigma_p(T) \cap \mathbb{T}. \end{cases}$$

Proof: Let

$$Y = \{(x, \lambda) \in X \times \mathbb{T} \colon \|x\| = 1,\ Tx = \lambda x\},$$

and define $\pi\colon Y \to \mathbb{T}$ by $\pi(x, \lambda) = \lambda$. Then Y is a complete, separable metric space, π is a continuous map, and $\pi(Y) = \sigma_p(T) \cap \mathbb{T}$. Therefore $\sigma_p(T) \cap \mathbb{T}$ is an analytic set.

Furthermore, by von Neumann's theorem cross-section theorem [16, Theorem 5.5.2], there exists a universally measurable function $s\colon \pi(Y) \to Y$ such that $\pi \circ s$ is the identity on $\pi(Y)$. Let g be the first coordinate of s. Then $g\colon \sigma_p(T) \cap \mathbb{T} \to X$ is a universally measurable function such that

$$Tg(\lambda) = \lambda g(\lambda) \quad \text{and} \quad \|g(\lambda)\| = 1 \quad (\lambda \in \sigma_p(T) \cap \mathbb{T}).$$

We extend g to the whole of $\mathbb{T}$ by setting it equal to zero elsewhere. ∎

Remark: Kaufman [10] has shown that, given any bounded, analytic subset A of $\mathbb{C}$, there exists an operator T on c_0 such that $\sigma_p(T) = A$. This explains the presence of analytic sets in the lemma above. Note, however, that if T is an operator on a *reflexive*, separable space, then $\sigma_p(T)$ is necessarily an F_σ-set [13].

3. Proofs of Theorems 1.1 and 1.2

The proofs of Theorems 1.1 and 1.2 are based upon two lemmas. The first of these is a simple inequality which enshrines the main idea of [7].

LEMMA 3.1: *Let X be a Banach space, let T be an operator on X, let $\alpha, \beta \in \sigma_p(T)$, and let $x, y \in X$ be norm-one eigenvectors corresponding to α, β respectively. Then*

$$|\alpha^n - \beta^n| \leq 2\|T^n\|\|x - y\| \quad (n \geq 1).$$

Proof: Fix $n \geq 1$ and consider $\|T^n x - T^n y\|$. On the one hand, we have

$$\begin{aligned} \|T^n x - T^n y\| &= \|\alpha^n x - \beta^n y\| \\ &\geq \|\alpha^n x - \beta^n x\| - \|\beta^n x - \beta^n y\| \\ &= |\alpha^n - \beta^n| - |\beta^n|\|x - y\|. \end{aligned}$$

On the other hand, evidently

$$\|T^n x - T^n x\| \leq \|T^n\|\|x - y\|.$$

The result follows upon combining these two inequalities and noting that $|\beta^n| \leq \|T^n\|$. ∎

LEMMA 3.2: *Let X be a separable Banach space, let T be an operator on X, and let μ be a Borel probability measure on $\mathbb{T}$ such that $\mu(\sigma_p(T) \cap \mathbb{T}) > 0$. Then, given $\epsilon > 0$, there exists a Borel probability measure ν on $\mathbb{T}$ such that $\nu \ll \mu$ and*

$$|\widehat{\nu}(n)|^2 \geq 1 - \epsilon\|T^n\| \quad (n \geq 1).$$

Proof: By Lemma 2.1, there exists a universally measurable function $g \colon \mathbb{T} \to X$ satisfying (1). By Lemma 3.1,

$$|\alpha^n - \beta^n| \leq 2\|T^n\|\|g(\alpha) - g(\beta)\| \quad (\alpha, \beta \in \sigma_p(T) \cap \mathbb{T},\ n \geq 1). \tag{2}$$

As X is separable, its unit sphere can be covered by a countable set of balls B_j of diameter $\epsilon/2$. Reducing ϵ, if necessary, we can suppose that none of these balls contains 0. The sets $g^{-1}(B_j)$ cover $\sigma_p(T) \cap \mathbb{T}$, so at least one of them satisfies $\mu(g^{-1}(B_j)) > 0$. Fix this j, and define a new Borel measure ν on $\mathbb{T}$ by

$$\nu(E) := \frac{\mu(E \cap g^{-1}(B_j))}{\mu(g^{-1}(B_j))}.$$

Then ν is probability measure concentrated on $g^{-1}(B_j)$, so, using (2), we have

$$\iint |e^{in\theta} - e^{in\phi}| d\nu(\theta) d\nu(\phi) \leq \epsilon\|T^n\| \quad (n \geq 1).$$

On the other hand, for each integer n, we have

$$\iint |e^{in\theta} - e^{in\phi}|d\nu(\theta)d\nu(\phi) \geq \left| \iint (1 - e^{in(\theta-\phi)})d\nu(\theta)d\nu(\phi) \right| = 1 - |\widehat{\nu}(n)|^2.$$

The result follows. ∎

Proof of Theorem 1.1: Assume that $\sigma_p(T) \cap \mathbb{T}$ is uncountable. Since it is also an analytic set, it contains a subset homeomorphic to the Cantor set [16, Theorem 4.3.5], and so it supports a continuous Borel probability measure, μ say.

We first show that, for each $k \geq 1$, the set

$$Z_k := \{n \geq 1 : \|T^n\| \leq k\}$$

is of density zero. By Lemma 3.2, applied with $\epsilon = 1/(2k)$, there exists a probability measure $\nu \ll \mu$ such that

$$|\widehat{\nu}(n)|^2 \geq 1/2 \quad (n \in Z_k).$$

Since μ is continuous, so too is ν, and therefore by a theorem of Wiener [9, p. 42, Corollary],

$$\frac{1}{N} \sum_{n=-N}^{N} |\widehat{\nu}(n)|^2 \to 0 \quad \text{as } N \to \infty.$$

It follows immediately that Z_k is of density zero, as claimed.

By definition of density zero, for each $k \geq 1$, there exists an integer m_k such that

$$\frac{1}{m} \operatorname{card}(Z_k \cap [1, m]) < \frac{1}{k} \quad (m \geq m_k).$$

We may suppose that the sequence (m_k) is strictly increasing. Define

$$Z := \bigcup_{k \geq 1} (Z_k \cap [m_k, m_{k+1})).$$

Then Z is a set of density zero, and $\lim_{\substack{n\to\infty \\ n \notin Z}} \|T^n\| = \infty$. This completes the proof. ∎

To prove Theorem 1.2, we also need an elementary lemma on tensor products.

LEMMA 3.3: *Let X be a Banach space, let T be an operator on X and let k, l be positive integers. Let $\overline{X}$ be the Banach space X with the scalar multiplication replaced by $(\lambda, x) \mapsto \overline{\lambda}x$, and let $\overline{T}$ be the linear operator on $\overline{X}$ defined by $\overline{T}x = Tx$ $(x \in \overline{X})$. Let*

$$Y = X\hat{\otimes} \cdots \hat{\otimes}X\hat{\otimes}\overline{X}\hat{\otimes} \cdots \hat{\otimes}\overline{X},$$

the projective tensor product of k copies of X and l copies of $\overline{X}$, and let

$$S = T \otimes \cdots \otimes T \otimes \overline{T} \otimes \cdots \otimes \overline{T},$$

the tensor product of k copies of T and l copies of $\overline{T}$. Then

$$\sigma_p(S) \supset \{z_1 \cdots z_k\overline{w}_1 \cdots \overline{w}_l \colon z_1, \ldots, z_k, w_1, \ldots, w_l \in \sigma_p(T)\}$$

and

$$\|S^n\| = \|T^n\|^{k+l} \quad (n \geq 1).$$

Proof: The statement about $\sigma_p(S)$ follows from the observation that, if $Tx_j = z_jx_j$ $(j = 1, \ldots, k)$ and $Ty_j = w_jy_j$ $(j = 1, \ldots, l)$, then

$$\begin{aligned} S(x_1 \otimes & \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_l) \\ &= (z_1 \cdots z_k\overline{w}_1 \cdots \overline{w}_l)(x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_l). \end{aligned}$$

The statement about $\|S^n\|$ is standard. ∎

Proof of Theorem 1.2: Assume that $\|T^n\| \not\to \infty$ as $n \to \infty$. Thus, there exist a constant M and a strictly increasing sequence of positive integers $(n_j)_{j\geq 1}$ such that $\|T^{n_j}\| \leq M$ for all $j \geq 1$.

We first show that $\sigma_p(T) \cap \mathbb{T}$ is a set of extended uniqueness. Assume the contrary. Then there exists a positive Rajchman measure μ on $\mathbb{T}$ such that $\mu(\sigma_p(T) \cap \mathbb{T}) > 0$. We may suppose that μ is a probability measure. By Lemma 3.2 applied with $\epsilon = 1/(2M)$, there exists a probability measure $\nu \ll \mu$ such that

$$|\widehat{\nu}(n_j)|^2 \geq 1/2 \quad (j \geq 1).$$

But since μ is a Rajchman measure, so too is ν [12, p. 77, Lemma 4], i.e., $\widehat{\nu}(n) \to 0$ as $|n| \to \infty$. This contradiction shows that $\sigma_p(T) \cap \mathbb{T}$ is indeed a set of extended uniqueness.

We now use this to deduce that the group generated by $\sigma_p(T) \cap \mathbb{T}$ is also a set of extended uniqueness. This group is equal to $\bigcup_{k,l\geq 1} E_{k,l}$, where

$$E_{k,l} = \{z_1 \cdots z_k\overline{w}_1 \cdots \overline{w}_l \colon z_1, \ldots, z_k, w_1, \ldots, w_l \in \sigma_p(T) \cap \mathbb{T}\}.$$

Note that the $E_{k,l}$ are analytic sets, hence universally measurable. It thus suffices to prove that each $E_{k,l}$ is a set of extended uniqueness. To do this, fix k, l, and define Y, S as in Lemma 3.3. Then Y is a separable Banach space, and S is an operator on Y such that $\sigma_p(S) \cap \mathbb{T} \supset E_{k,l}$ and $\|S^n\| \not\to \infty$ as $n \to \infty$. From what we have already proved, $\sigma_p(S) \cap \mathbb{T}$ is a set of extended uniqueness, and therefore so too is $E_{k,l}$, as desired. ∎

Remark: There certainly do exist uncountable analytic subgroups of $\mathbb{T}$ which are sets of extended uniqueness. Examples include the so-called H_2-groups as described in [6, §4]. Interestingly, some of these same groups occur as the point spectra of shift operators in ergodic theory. Though the operators in question act on L^∞-spaces, which are not separable, perhaps they might eventually be adapted to construct examples for which $\|T^n\| \not\to \infty$ in Theorem 1.1.

4. Proofs of Theorems 1.6 and 1.9

The proofs of Theorems 1.6 and 1.9 are based on the following lemma, which was inspired by ideas from [14].

LEMMA 4.1: *Let X be a separable Banach space, let T be an operator on X, and let μ be a Borel probability measure on $\mathbb{T}$ such that $\mu(\sigma_p(T) \cap \mathbb{T}) > 0$. Then there exists a bounded, universally measurable function $h\colon \mathbb{T} \to X$ such that*

$$\frac{1}{\|T^n\|} \leq \left\| \int h(e^{i\theta}) e^{-in\theta} d\mu(\theta) \right\| \quad (n \geq 1). \tag{3}$$

Proof: By Lemma 2.1, there exists a universally measurable function $g\colon \mathbb{T} \to X$ satisfying (1). Let $(x_n)_{n \in \mathbb{Z}}$ be the sequence of vectors in X defined by

$$x_n = \int g(e^{i\theta}) e^{-in\theta} d\mu(\theta) \quad (n \in \mathbb{Z}).$$

We first claim that there exists $n_0 \in \mathbb{Z}$ such that $x_{n_0} \neq 0$. For suppose, to the contrary, that $x_n = 0$ for all $n \in \mathbb{Z}$. Then, given a linear functional $\psi \in X^*$, we have

$$0 = \psi(x_n) = \int e^{-in\theta} \psi(g(e^{i\theta})) d\mu(\theta) \quad (n \in \mathbb{Z}),$$

i.e., the Fourier coefficients of the measure $\psi(g(e^{i\theta}))d\mu(\theta)$ are all zero. It follows that $\psi(g(e^{i\theta})) = 0$ μ-almost everywhere. As X is separable, countably many ψ suffice to separate points, and so $g(e^{i\theta}) = 0$ μ-almost everywhere. This contradicts the hypothesis that $\mu(\sigma_p(T) \cap \mathbb{T}) > 0$.

Next, we calculate the action of T on the sequence (x_n). By standard properties of the Bochner integral, together with (1), we have

$$Tx_n = \int Tg(e^{i\theta})e^{-in\theta}d\mu(\theta) = \int e^{i\theta}g(e^{i\theta})e^{-in\theta}d\mu(\theta) = x_{n-1} \quad (n \in \mathbb{Z}).$$

In particular, $T^n(x_{n_0+n}) = x_{n_0}$ for all $n \geq 1$. It follows that

$$\frac{1}{\|T^n\|} \leq \frac{\|x_{n_0+n}\|}{\|x_{n_0}\|} = \frac{1}{\|x_{n_0}\|}\left\| \int g(e^{i\theta})e^{-i(n_0+n)\theta}d\mu(\theta)\right\| \quad (n \geq 1).$$

Define $h\colon \mathbb{T} \to X$ by $h(e^{i\theta}) := g(e^{i\theta})e^{-in_0\theta}/\|x_{n_0}\|$. Then h satisfies the conclusion of the lemma. ∎

Proof of Theorem 1.6: By hypothesis $m(\sigma_p(T) \cap \mathbb{T}) > 0$, where m denotes normalized Lebesgue measure on $\mathbb{T}$. Hence, by Lemma 4.1, there exists a bounded, universally measurable function $h\colon \mathbb{T} \to X$ such that

$$\frac{1}{\|T^n\|} \leq \left\| \int h(e^{i\theta})e^{-in\theta}dm(\theta)\right\| = \|\widehat{h}(n)\| \quad (n \geq 1).$$

Since X is of Fourier type p and $h \in L^p(\mathbb{T}, X)$, we have

$$\sum_{n\in\mathbb{Z}} \|\widehat{h}(n)\|^q < \infty.$$

The result follows upon combining these inequalities. ∎

For the proof of Theorem 1.9, we need an analogue of the notion of Fourier type p for measures μ other than Lebesgue measure. We shall prove an inequality for 'μ-Fourier coefficients' in terms of $I_\Phi(\mu)$, the Φ-energy of μ, which was defined in §1.

Given a Banach space X (not necessarily separable) and a Borel probability measure μ on $\mathbb{T}$, we write $L^p(\mu, X)$ for the set of all Bochner-integrable functions $f\colon (\mathbb{T}, \mu) \to X$ such that $\|f\|_p < \infty$, where

$$\|f\|_p := \begin{cases} (\int \|f(e^{i\theta})\|^p d\mu(\theta))^{1/p}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_\mu \|f(e^{i\theta})\|, & p = \infty. \end{cases}$$

We first prove an inequality for Hilbert spaces (this is implicit in [14]).

LEMMA 4.2: *Let Φ be a kernel, and let μ be a Borel probability measure on $\mathbb{T}$ such that $I_\Phi(\mu) < \infty$. Let H be a Hilbert space, and let $f \in L^\infty(\mu, H)$. Then*

$$\sum_{n\in\mathbb{Z}} \widehat{\Phi}(n)\left\| \int f(e^{i\theta})e^{-in\theta}d\mu(\theta)\right\|^2 \leq I_\Phi(\mu)\|f\|^2_{L^\infty(\mu,H)}.$$

Proof: Suppose first that $\sum_{n\in\mathbb{Z}}\widehat{\Phi}(n)<\infty$. Writing $\langle\cdot,\cdot\rangle$ for the inner product on H, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}}\widehat{\Phi}(n)\left\|\int f(e^{i\theta})e^{-in\theta}d\mu(\theta)\right\|^2\\
&\qquad=\sum_{n\in\mathbb{Z}}\widehat{\Phi}(n)\left\langle\int f(e^{i\theta})e^{-in\theta}d\mu(\theta),\int f(e^{i\phi})e^{-in\phi}d\mu(\phi)\right\rangle\\
&\qquad=\sum_{n\in\mathbb{Z}}\widehat{\Phi}(n)\iint\langle f(e^{i\theta}),f(e^{i\phi})\rangle e^{in(\phi-\theta)}d\mu(\theta)d\mu(\phi)\\
&\qquad=\iint\langle f(e^{i\theta}),f(e^{i\phi})\rangle\Phi(\phi-\theta)d\mu(\theta)d\mu(\phi)\\
&\qquad\le\|f\|^2_{L^\infty(\mu,H)}I_\Phi(\mu).
\end{aligned}$$

The general case follows by approximating Φ, exactly as in [8, p. 35, Proposition 3]. ∎

We extend this to more general Banach spaces using interpolation.

LEMMA 4.3: *Let Φ be a kernel, and let μ be a Borel probability measure on $\mathbb{T}$ such that $I_\Phi(\mu)<\infty$. Let $1<p\le 2$ and $1/p+1/q=1$, and let X be a Banach space which is $2/q$-Hilbertian. Let $f\in L^\infty(\mu,X)$. Then*

$$\sum_{n\in\mathbb{Z}}\widehat{\Phi}(n)^{q/2}\left\|\int f(e^{i\theta})e^{-in\theta}d\mu(\theta)\right\|^q\le\sup_{n\in\mathbb{Z}}\widehat{\Phi}(n)^{q/2-1}I_\Phi(\mu)^{1/2}\|f\|^q_{L^\infty(\mu,X)}.$$

Proof: By hypothesis, $X=[Y,H]_{2/q}$, where H is a Hilbert space and Y is a Banach space.

Given a vector-valued, Bochner-integrable function f on $(\mathbb{T},\mu)$, let $R(f)$ denote the sequence of vectors

$$R(f):=\left(\widehat{\Phi}(n)^{1/2}\int f(e^{i\theta})e^{-in\theta}d\mu(\theta)\right)_{n\in\mathbb{Z}}.$$

Clearly, R is a bounded linear map from $L^\infty(\mu,Y)$ to $\ell^\infty(\mathbb{Z},Y)$ with

$$\|R\colon L^\infty(\mu,Y)\to\ell^\infty(\mathbb{Z},Y)\|\le\sup_{n\in\mathbb{Z}}\widehat{\Phi}(n)^{1/2}.$$

Further, by Lemma 4.2, R is a bounded linear map from $L^\infty(\mu,H)$ to $\ell^2(\mathbb{Z},H)$ with

$$\|R\colon L^\infty(\mu,H)\to\ell^2(\mathbb{Z},H)\|\le I_\Phi(\mu)^{1/2}.$$

By the interpolation theorem [1, Theorem 4.1.2], it follows that R is also a bounded linear map from $[L^\infty(\mu,Y),L^\infty(\mu,H)]_{2/q}$ to $[\ell^\infty(\mathbb{Z},Y),\ell^2(\mathbb{Z},H)]_{2/q}$, with norm at most

$$\sup_{n\in\mathbb{Z}}\widehat{\Phi}(n)^{1/2-1/q}I_\Phi(\mu)^{1/q}.$$

Now $X=[Y,H]_{2/q}$, so it follows from the definition of the complex interpolation method that $L^\infty(\mu,X)\subset[L^\infty(\mu,Y),L^\infty(\mu,H)]_{2/q}$, the inclusion being an isometry. Also, by [1, Theorem 5.1.2], we have $[\ell^\infty(\mathbb{Z},Y),\ell^2(\mathbb{Z},H)]_{2/q}=\ell^q(\mathbb{Z},X)$, with equality of norms. Hence, R is a bounded linear map from $L^\infty(\mu,X)$ to $\ell^q(\mathbb{Z},X)$ with

$$\|R\colon L^\infty(\mu,X)\to\ell^q(\mathbb{Z},X)\|\le\sup_{n\in\mathbb{Z}}\widehat{\Phi}(n)^{1/2-1/q}I_\Phi(\mu)^{1/q}.$$

This completes the proof. ∎

Proof of Theorem 1.9: Since $\sigma_p(T)\cap\mathbb{T}$ is of positive Φ-capacity, there exists a Borel probability measure μ on $\sigma_p(T)\cap\mathbb{T}$ such that $I_\Phi(\mu)<\infty$. By Lemma 4.1, there exists a bounded, universally measurable function $h\colon\mathbb{T}\to X$ such that

$$\frac{1}{\|T^n\|}\le\left\|\int h(e^{i\theta})e^{-in\theta}\,d\mu(\theta)\right\|\quad(n\ge1).$$

As X is separable, $h\in L^\infty(\mu,X)$, so by Lemma 4.3,

$$\sum_{n\in\mathbb{Z}}\widehat{\Phi}(n)^{q/2}\left\|\int h(e^{i\theta})e^{-in\theta}\,d\mu(\theta)\right\|^q<\infty.$$

The result follows upon combining these inequalities. ∎

5. Construction of the examples

To construct Example 1.5, we need the following elementary lemma. A sequence $(\omega)_{n\ge1}$ is called **submultiplicative** if $\omega_{m+n}\le\omega_m\omega_n$ for all $m,n\ge1$.

LEMMA 5.1: *Let $(\gamma)_{n\ge1}$ be a sequence such that $\gamma_n>1$ for all n and $\lim_{n\to\infty}\gamma_n=\infty$. Then there exists a submultiplicative sequence $(\omega_n)_{n\ge1}$ such that $\omega_n\le\gamma_n$ for all n and $\lim_{n\to\infty}\omega_n=\infty$.*

Proof: Define

$$\omega_n:=\min\{\gamma_{n_1}\cdots\gamma_{n_k}\colon n_1+\cdots+n_k=n\}\quad(n\ge1).$$

Clearly (ω_n) is submultiplicative and $\omega_n \le \gamma_n$ for all n. It remains to show that $\lim_{n\to\infty} \omega_n = \infty$. For this, observe that if $n_1 + \cdots + n_k = n$ and $n \ge m^2$, then either $k \ge m$ or $\max_{1\le j\le k} n_j \ge m$. Hence

$$\min_{n\ge m^2} \omega_n \ge \min(\min_{n\ge m} \gamma_n, (\min_{n\ge 1} \gamma_n)^m) \quad (m \ge 1).$$

The right-hand side tends to infinity as $m \to \infty$, hence so too does the left-hand side. ∎

Construction of Example 1.5: Let $(\omega_n)_{n\ge1}$ be the sequence furnished by Lemma 5.1, and extend it to $n = 0$ by defining $\omega_0 = 1$. Let $(X, \|\cdot\|)$ be the Banach space defined by

$$X := \{(x_n)_{n\ge0} \colon x_n/\omega_n \to 0\}, \quad \|(x_n)\| := \sup_{n\ge0} |x_n|/\omega_n.$$

If S denotes the left shift, then S is a bounded operator on X with $\|S^n\| = \omega_n \le \gamma_n$ $(n \ge 1)$. Also, for each $\lambda \in \mathbb{T}$, the vector $x = (1, \lambda, \lambda^2, \ldots)$ is an eigenvector of S with eigenvalue λ, so $\sigma_p(S) \cap \mathbb{T} = \mathbb{T}$. Finally, we remark that the space X is isometrically isomorphic to c_0, so S is similar to an operator T on c_0 with the required properties. ∎

For Example 1.7, we need an ℓ^q-analogue of Lemma 5.1.

LEMMA 5.2: *Let $q \ge 1$ and let $(\gamma)_{n\ge1}$ be a sequence such that $\gamma_n > 1$ for all n and $\sum_n \gamma_n^{-q} < \infty$. Then there is a submultiplicative sequence $(\omega_n)_{n\ge1}$ such that $\omega_n \le \gamma_n$ for all n and $\sum_n \omega_n^{-q} < \infty$.*

Proof: Define ω_n as in the proof of Lemma 5.1. All that remains to be shown is that $\sum_n \omega_n^{-q} < \infty$. For this, note that

$$\omega_n^{-q} = \max\{\gamma_{n_1}^{-q} \cdots \gamma_{n_k}^{-q} \colon n_1 + \cdots + n_k = n\} \le \sum_{n_1+\cdots+n_k=n} \gamma_{n_1}^{-q} \cdots \gamma_{n_k}^{-q}.$$

Hence

$$\sum_{n\ge1} \omega_n^{-q} \le \prod_{l\ge1} (1 - \gamma_l^{-q})^{-1},$$

and the infinite product converges because $\sum_n \gamma_n^{-q} < \infty$. ∎

Construction of Example 1.7: We repeat the construction of Example 1.5, with $(\omega_n)_{n\ge1}$ now given by Lemma 5.2, $\omega_0 = 1$, and

$$X := \left\{ (x_n)_{n\ge0} \colon \|(x_n)\| := \left(\sum_{n\ge0} |x_n|^q/\omega_n^q \right)^{1/q} < \infty \right\}.$$

The left shift S still satisfies $\|S^n\| = \omega_n \leq \gamma_n$ $(n \geq 1)$ and $\sigma_p(S) \cap \mathbb{T} = \mathbb{T}$. As X is isometrically isomorphic to ℓ^q, the operator S is similar to an operator T on ℓ^q with the required properties. ∎

For the remaining two examples, we need the following result from [2]. Given a subset E of $\mathbb{T}$, we write E_δ for the set of points of $\mathbb{T}$ whose arclength distance from E is at most δ, and $|E_\delta|$ for the Lebesgue measure of this set.

LEMMA 5.3 ([2, Theorem 1.2]): *Let E be a closed subset of $\mathbb{T}$ and let (η_n) be a positive sequence such that $\lim_{n\to\infty} \eta_n = \infty$. Then there exists an operator T on ℓ^2 such that $\sigma_p(T) = E$ and*

$$\sum_n \eta_n |E_{1/n}| \|T^n\|^{-2} = \infty. \qquad ∎$$

Construction of Example 1.8: If E is a closed subset of $\mathbb{T}$ of Lebesgue measure zero, then $|E_{1/n}| \to 0$ as $n \to \infty$. Consequently, we can take $\eta_n := 1/|E_{1/n}|$ in Lemma 5.3 to obtain an operator T on ℓ^2 such that $\sigma_p(T) = E$ and $\sum_n \|T^n\|^{-2} = \infty$. ∎

Construction of Example 1.12: Choose $\alpha' < \alpha$ such that the upper Minkowski dimension of E is less than α'. By [4, Proposition 3.2], we have $|E_{1/n}| = O(n^{\alpha'-1})$ as $n \to \infty$. Taking $\eta_n := n^{\alpha-\alpha'}$ in Lemma 5.3, we obtain an operator T on ℓ^2 such that $\sigma_p(T) = E$ and $\sum_n n^{\alpha-1}\|T^n\|^{-2} = \infty$. ∎

6. Continuous semigroups of operators

As often happens, our results on operator powers $(T^n)_{n\geq 1}$ have analogues for continuous semigroups of operators $(T_t)_{t\geq 0}$. In this instance, the continuous versions are actually a direct consequence of the discrete versions. The purpose of this section is to outline the details.

We begin recalling some standard definitions. Let X be a complex Banach space. A family of bounded operators $(T_t)_{t\geq 0}$ on X is called a **C_0-semigroup** if

- $T_0 = I$,
- $T_s T_t = T_{s+t}$ for all $s, t \geq 0$, and
- $\lim_{t\to 0^+} \|T_t x - x\| = 0$ for all $x \in X$.

The map $(t, x) \mapsto T_t x$: $[0, \infty) \times X \to X$ is then continuous, and there exist constants M, ω such that $\|T_t\| \leq M e^{\omega t}$ $(t \geq 0)$.

The **generator** of the C_0-semigroup $(T_t)_{t\geq 0}$ is the map $A\colon D(A) \to X$ defined by

$$D(A) := \Big\{x \in X\colon \lim_{t\to 0+} \frac{T_t x - x}{t} \text{ exists}\Big\},$$
$$Ax := \lim_{t\to 0+} \frac{T_t x - x}{t} \quad (x \in D(A)).$$

The generator A is always a closed, densely defined operator. It generates the semigroup in the sense that $T_t = e^{tA}$ $(t \geq 0)$ (appropriately interpreted). Also, $\sigma_p(T_t) \setminus \{0\} = \exp(t\sigma_p(A))$ $(t \geq 0)$. In particular, this implies that

$$\sigma_p(T_1) \cap \mathbb{T} = \exp(\sigma_p(A) \cap i\mathbb{R}). \tag{4}$$

For further details we refer to [3], for example.

All of the theorems that follow depend on one very simple fact. We write $\lceil t \rceil$ for the smallest integer n such that $n \geq t$.

LEMMA 6.1: *Let $(T_t)_{t\geq 0}$ be a C_0-semigroup. Set $C = \sup_{s\in[0,1]} \|T_s\|$. Then*

$$\|T_t\| \geq C^{-1} \|T_1^{\lceil t \rceil}\| \quad (t \geq 0). \tag{5}$$

Proof: Write $t = \lceil t \rceil - s$, where $s \in [0,1)$. Then

$$\|T_1^{\lceil t \rceil}\| = \|T_{\lceil t \rceil}\| = \|T_{s+t}\| \leq \|T_s\|\|T_t\| \leq C\|T_t\|. \qquad \blacksquare$$

Our first result is a C_0-semigroup analogue of Theorem 1.1. Let us say that a Borel subset Z of $\mathbb{R}^+$ is of **Lebesgue density zero** if $|Z \cap [0,t]|/t \to 0$ as $t \to \infty$.

THEOREM 6.2: *Let X be a separable Banach space, and let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X with generator A such that $\sigma_p(A) \cap i\mathbb{R}$ is uncountable. Then there exists a subset Z of $\mathbb{R}^+$ of Lebesgue density zero such that*

$$\lim_{\substack{t\to\infty \\ t\notin Z}} \|T_t\| = \infty.$$

Proof: From (4) it follows that $\sigma_p(T_1) \cap \mathbb{T}$ is uncountable. Hence, by Theorem 1.1, there exists a set Z' of positive integers such that Z' is of density zero and $\|T_1^n\| \to \infty$ as $n \to \infty$, $n \notin Z'$. Set $Z := \bigcup_{n\in Z'} (n-1, n]$. Then Z is of Lebesgue density zero and, thanks to (5), we have $\|T_t\| \to \infty$ as $t \to \infty$, $t \notin Z$. $\blacksquare$

Next, we prove an analogue of Theorem 1.2.

THEOREM 6.3: *Let X be a separable Banach space, and let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X with generator A. If $\|T_t\| \not\to \infty$ as $t \to \infty$, then $\sigma_p(A) \cap i\mathbb{R}$ is contained in a proper subgroup of $i\mathbb{R}$.*

Proof: Assume that $\|T_t\| \not\to \infty$ as $t \to \infty$. From Lemma 6.1, it follows that $\|T_1^n\| \not\to \infty$ as $n \to \infty$. By Theorem 1.2, $\sigma_p(T) \cap \mathbb{T}$ is contained in a proper subgroup of $\mathbb{T}$. Using (4), this implies that $\sigma_p(A) \cap i\mathbb{R}$ is contained in a proper subgroup of $i\mathbb{R}$. ∎

The following result is an analogue of Theorem 1.6.

THEOREM 6.4: *Let $1 < p \leq 2$ and $1/p + 1/q = 1$. Let X be a separable Banach space of Fourier type p. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X with generator A, and suppose that $\sigma_p(A) \cap i\mathbb{R}$ is of positive Lebesgue measure. Then*

$$\int_0^\infty \|T_t\|^{-q} dt < \infty.$$

Proof: From (4), we see that $\sigma_p(T) \cap \mathbb{T}$ is of positive Lebesgue measure. Hence, by Theorem 1.6, we have $\sum_n \|T_1^n\|^{-q} < \infty$. Using (5), it follows that

$$\int_0^\infty \|T_t\|^{-q} dt = \sum_{n\geq 1} \int_{n-1}^n \|T_t\|^{-q} dt \leq \sum_{n\geq 1} C^{-1} \|T_1^n\|^{-q} < \infty. \quad ∎$$

Finally, here are C_0-semigroup versions of Corollaries 1.10 and 1.11. The proofs are entirely analogous to that of Theorem 6.4, and are therefore omitted.

THEOREM 6.5: *Let X be a separable Hilbert space. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X with generator A, and suppose that $\sigma_p(A) \cap i\mathbb{R}$ is of positive logarithmic capacity. Then*

$$\int_1^\infty t^{-1} \|T_t\|^{-2} dt < \infty. \quad ∎$$

THEOREM 6.6: *Let $1 < p \leq 2$ and $1/p + 1/q = 1$, and let $\alpha \geq 0$. Let X be a separable Banach space which is $2/q$-Hilbertian. Let $(T_t)_{t\geq 0}$ be a C_0-semigroup on X with generator A, and suppose that $\sigma_p(A) \cap i\mathbb{R}$ is of Hausdorff dimension $> \alpha$. Then*

$$\int_1^\infty t^{(\alpha-1)q/2} \|T_t\|^{-q} dt < \infty. \quad ∎$$

ACKNOWLEDGEMENT: The author is grateful to Stephen Dilworth, Russell Lyons and Jaroslav Zemánek for drawing his attention to [15], [6] and [7] respectively.

References

[1] J. Bergh and J. Löfstrom, *Interpolation Spaces*, Springer-Verlag, Berlin, 1976.

[2] O. El-Fallah and T. J. Ransford, *Peripheral point spectrum and growth of powers of operators*, Journal of Operator Theory **52** (2004), 89–101.

[3] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New York, 2000.

[4] K. Falconer, *Fractal Geometry*, Wiley, Chichester, 1990.

[5] Kh. Garsia-Kuerva, K. S. Kazaryan, V. I. Kolyada and Kh. L. Torrea, *The Hausdorff–Young inequality with vector-valued coefficients and applications*, Russian Mathematical Surveys **53** (1998), 435–513.

[6] B. Host, J.-F. Méla and F. Parreau, *Non singular transformations and spectral analysis of measures*, Bulletin de la Société Mathématique de France **119** (1991), 33–90.

[7] B. Jamison, *Eigenvalues of modulus* 1, Proceedings of the American Mathematical Society **16** (1965), 375–377.

[8] J.-P. Kahane and R. Salem, *Ensembles parfaits et séries trigonométriques*, 2e édition, Hermann, Paris, 1994.

[9] Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover, New York, 1976.

[10] R. Kaufman, *On some operators in* c_0, Israel Journal of Mathematics **50** (1985), 353–356.

[11] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, New York, 1995.

[12] A. S. Kechris and A. Louveau, *Descriptive Set Theory and the Structure of Sets of Uniqueness*, Cambridge University Press, Cambridge, 1987.

[13] L. N. Nikolskaja, *Structure of the point spectrum of a linear operator*, Mathematical Notes **15** (1974), 83–87.

[14] N. K. Nikolskii, *Selected problems of weighted approximation and spectral analysis*, Proceedings of the Steklov Institute of Mathematics **120** (1974), American Mathematical Society, Providence, RI, 1976.

[15] G. Pisier, *Some applications of the complex interpolation method to Banach lattices*, Journal d'Analyse Mathématique **35** (1979), 264–281.

[16] S. M. Srivastava, *A Course on Borel Sets*, Springer-Verlag, New York, 1998.